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Analytic Solution of Gauge Field Equations for Lorentz Gravity

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The exterior analytic solution for a static, spherically symmetric system is given by means of a set of gauge field equations from Lorentz gravity in the curvature coordinate. The correction contributed by the gravitational gauge field in the exterior of a static sphere is obtained for the gravity.

1. INTRODUCTION

Massouri and Chang (1976) obtained the field equations of Lorentz gravity

$$R^i_\mu - V^i_\mu R = -CT^i_\mu - \rho t^i_\mu \tag{1}$$

$$CS^\mu_{ij} + K^\mu_{ij} = -\rho F^{\mu\nu}_{ij\|\nu} \tag{2}$$

which were all given by Shao and Xu (1986). In the above equations, $i = 0, 1, 2, 3$ is the moving frame index, i.e., Lorentz index; Greek letters μ, $\nu, \ldots$ are the natural frame indices on the spacetime manifold M; ρ is the gauge gravitational constant; $C = 8\pi k$ (k is the Newtonian gravitational constant); R is the Einstein curvature scalar on M; and R^i_μ is the Ricci tensor in the moving frame.

T^i_μ is the mass tensor in the moving frame, and t^i_μ is the energy-momentum tensor of gauge field in the same frame as above, where

$$T^i_\mu = \frac{1}{2V}\frac{d(\mathscr{L}_m V)}{\delta V^\mu_i} \tag{3}$$

$$t^i_\mu = -\mathrm{tr}(F_{\mu\sigma}F^{\lambda\sigma})V^i_\lambda + \tfrac{1}{4}\,\mathrm{tr}(F_{\lambda\sigma}F^{\lambda\sigma})V^i_\mu \tag{4}$$

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$\mathscr{L}_m = \mathscr{L}_m(\psi, \psi, \mu)$ is the Lagrangian of the matter field ψ, $V = \det(V^i_\mu)$ and V^i_μ are the Lorentz vierbein fields, which are gauge potentials in the gauge theory of gravity (GTG). We have

$$F_{\mu\nu} = (F_{\mu\nu}{}^i{}_j) \tag{5}$$

with

$$F_{\mu\nu}{}^i{}_j = \partial_\mu B^i_{\nu j} - \partial_\nu B^i_{\mu j} + B^i_{\mu k} B^k_{\nu j} - B^i_{\nu k} B^k_{\mu j} \tag{6}$$

Here $F_{\mu\nu}{}^i{}_j$ is the curvature tensor of M in the moving frame and it is also the gauge field strength in GTG, $B^i_{\mu j}$ are the coefficients of the Lorentz connection and they correspond to the gauge potentials in GTG.

K^μ_{ij} and S^μ_{ij} in (2) are cotorsion and the spin current of the matter field ψ, respectively, and

$$K^\mu_{ij} \equiv Q^\mu_{ij} - Q^\lambda_{ki} V^k_\lambda V^\mu_j - Q^\lambda_{jk} V^k_\lambda V^\mu_i \tag{7}$$

$$S^\mu_{ij} \equiv \frac{1}{V} \frac{\partial(\mathscr{L}_m V)}{\partial B^{ij}_\mu} \tag{8}$$

where

$$Q^\mu_{ij} = (V^k_{\nu,\lambda} - V^k_{\lambda,\nu} + B^k_{\lambda\nu} - B^k_{\nu\lambda}) V^\nu_i V^\lambda_j V^\mu_k \tag{9}$$

is the torsion tensor of M in the moving frame. The "$\|$" in (2) denotes the twofold covariant derivative in the natural and moving frames, and

$$F^{\mu\nu}_{ij\|\nu} = F^{\mu\nu}_{ij,\nu} - B^k_{\nu i} F^{\mu\nu}_{kj} - B^k_{\nu j} F^{\mu\nu}_{ik} - \begin{Bmatrix} \lambda \\ \lambda\sigma \end{Bmatrix} F^{\mu\sigma}_{ij} - \begin{Bmatrix} \mu \\ \sigma\nu \end{Bmatrix} F^{\sigma\nu}_{ij} \tag{10}$$

If the spacetime manifold M is a Riemann space (torsion free), we write the gauge equations (1) and (2) in the natural frame as

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = -(C T_{\mu\nu} + \rho t_{\mu\nu}) \tag{11}$$

$$C S^\mu_{\nu\lambda} = -\rho F^{\mu\sigma}_{\nu\lambda\|\sigma} \tag{12}$$

and expression (4) becomes

$$t_{\mu\nu} = -\operatorname{tr}(R_{\mu\sigma} R^{\lambda\sigma}) g_{\lambda\nu} + \tfrac{1}{4} \operatorname{tr}(R_{\lambda\sigma} R^{\lambda\sigma}) g_{\mu\nu} \tag{13}$$

In this paper, we give the exterior solution of the above equations for a static, symmetric system (provided $\rho = 1$).

2. THE SOLUTION OF FIELD EQUATIONS

In order to obtain the exterior solution for (11) and (12) in static and spherically symmetric spacetime, we can write (11) and (12) as

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = -t_{\mu\nu} \tag{14}$$

$$F^{\mu\sigma}_{\nu\lambda\|\sigma} = 0 \tag{15}$$

where $\rho = 1$ is supposed, and let

$$g_{\mu\nu} = \mathrm{diag}(-B, A, r^2, r^2 \sin^2 \theta)$$

Here A and B are components of the spacetime metric to be solved and are functions of r. Using the above metric, we find the nonvanishing components of the Riemann tensor as follows:

$$R_{1212} = \tfrac{1}{4}A^{-1}B^{-1}(2ABB'' - AB'^2 - A'BB')$$
$$R_{1313} = \tfrac{1}{2}A^{-1}B'r, \qquad R_{1414} = \tfrac{1}{2}A^{-1}B'r\sin^2\theta$$
$$R_{2323} = \tfrac{1}{2}A^{-1}A'r, \qquad R_{2424} = \tfrac{1}{2}A^{-1}A'r\sin^2\theta$$
$$R_{3434} = A^{-1}(A-1)r^2\sin^2\theta$$

where

$$A' = \frac{dA}{dr}, \qquad B' = \frac{dB}{dr}$$

From the above expressions of the Riemann tensor, $R_{\mu\nu}$, R can be given, and then substituting them into (13), we can obtain $t_{\mu\nu}$. With the help of a computer, (14) yields

$$\begin{aligned}
&(4A^2B^2B''^2 - 4A^2BB'^2B'' - 4AA'B^2B'B'' + A^2B'^4 \\
&\quad + 2AA'BB'^3 + A'^2B^2B'^2)r^4 + 32A^2A'B^4r^3 \\
&\quad + (8A^2B^2B'^2 - 8A'^2B^4 + 32A^4B^4 - 32A^3B^4)r^2 \\
&\quad - 16A^4B^4 + 32A^3B^4 - 16A^2B^4 = 0
\end{aligned}$$

$$\begin{aligned}
&(4A^2B^2B''^2 - 4A^2BB'^2B'' - 4AA'B^2B'B'' + A^2B'^4 \\
&\quad + 2AA'BB'^3 + A'^2B^2B'^2)r^4 - 32A^3B^3B'r^3 \\
&\quad + (8A'^2B^4 + 32A^4B^4 - 32A^3B^4 - 8A^2B^2B'^2)r^2 \\
&\quad - 16A^4B^4 + 32A^3B^4 - 16A^2B^4 = 0
\end{aligned}$$

$$\begin{aligned}
&(4A^2B^2B''^2 - 4A^2BB'^2B'' - 4AA'B^2B'B'' + 16A^3B^3B'' \\
&\quad + A^2B'^4 + 2AA'BB'^3 + A'^2B^2B'^2 - 8A^3B^2B'^2 \\
&\quad - 8A^2A'B^3B')r^4 + (16A^3B^3B' - 16A^2A'B^4)r^3 \\
&\quad - 16A^4B^4 + 32A^3B^4 - 16A^2B^4 = 0
\end{aligned}$$

Only two of the above three equations are independent. Simplifying them, we obtain

$$A'B + AB' = 0 \tag{16}$$

$$(AB'^2 + A'BB' - 2ABB'')r^2 + 4(A'B^2 - ABB')r + 4(A^2B^2 - AB^2) = 0 \tag{17}$$

Via (16), we easily have

$$AB = \lambda$$

(λ is a constant independent of r). For convenience, let $y = A^{-1}$, $B = \lambda y$ in (17), and notice

$$A' = -y^{-2}y', \qquad B' = \lambda y', \qquad B'' = \lambda y''$$

After rearranging terms in (17), we find

$$r^2 y'' + 4ry' - 2(1-y) = 0$$

From the differential equation,

$$y = 1 + \frac{\varepsilon_1}{r} + \frac{\varepsilon_2}{\gamma^2}$$

where ε_1, ε_2 are constants. Therefore, the solution of equation (14) in the curvature coordinates can be written as

$$A = \left(1 + \frac{\varepsilon_1}{r} + \frac{\varepsilon_2}{r^2}\right)^{-1}, \qquad B = \lambda\left(1 + \frac{\varepsilon_1}{r} + \frac{\varepsilon_2}{r^2}\right) \tag{18}$$

Obviously, when $r \to \infty$, spacetime should be asymptotically flat, and thus $\lambda = 1$. Using (18) and components of the Riemann tensor as given previously in this section, equation (15) can be satisfied by means of the formulation of (10) considering $B^i_{\mu j}$ a the coefficients of the Lorentz connection.

In addition, when r is very large, the third term in (18) will be very small in comparison with the second term, and A, B must agree with the Schwarzschild metric, so that

$$\varepsilon_1 = -2km/c^2$$

where k is Newton's gravitational constant and m is the total mass of the spherical body producing the gravitational field, and c is the velocity of light. For convenient comparison, we let $\varepsilon_2 = \chi m$, replacing ε_2 with χ, another constant. We therefore obtain the analytic solution for the gauge field equations (14), (15) in the curvature coordinates:

$$g_{\mu\nu} = \mathrm{diag}\left(-\left(1 - \frac{2km}{r} + \frac{\chi m}{r^2}\right), \left(1 - \frac{2km}{r} + \frac{\chi m}{r^2}\right)^{-1}, r^2, r^2 \sin^2\theta\right) \tag{19}$$

3. CONCLUSION

(i) The result of this paper shows that the existence of the gauge field of gravitation will affect the metric of spacetime. Equation (19) gives the gauge correction due to the gauge field of gravitation to the Schwarzschild metric solution.

(ii) The term $\chi m/r^2$ in (19) reflects the effect of the gauge field of gravitation on the metric of spacetime. χ will be related to the coupling constant ρ.

(iii) We may use the correction to the result to explain the difference between observed and theoretical results in the tests of general relativity, and this may lead to a way of determining the constant χ.

(iv) Because (19) is an exact expression and the term $\chi m/r^2$ will make the variation of $g_{\mu\nu}$ with r different from that of the Schwarzschild metric, to a certain extent, GTG may be used to explain the existence of some "fifth force."

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